Introduction

These notes were written to accompany my ten lectures on tight closure at the Institute for Studies in Theoretical Physics and Mathematics (IPM), School of Mathematics, in Tehran, Iran, in January 2002. The participants ranged from researchers with knowledge in tight closure to graduate students new to tight closure and research in general. My goal was to introduce the theory of tight closure and show its utility and beauty through both its early and its recent applications. I worked through some more technical aspects of tight closure, for example test elements, because the theory of test elements is central to many applications, and furthermore, because it uses and develops some beautiful commutative algebra. Throughout I tried to emphasize the beauty of the commutative algebra as developed through tight closure, as well as the need to understand various aspects of commutative algebra to work in the area of tight closure (such as the need to understand excellent rings, Cohen-Macaulay rings, asymptotic properties of (Frobenius) powers of ideals, local cohomology, homology theory, etc.).

All this of course points to the depth and the acuity of the two originators of the theory, namely of Melvin Hochster and Craig Huneke. They started the theory in the mid 1980s. They took several standing proofs in commutative algebra, such as the proofs of the homological conjectures-theorems, the proof of Hochster and Roberts [HR1] that the ring of invariants of a reductive group acting on a polynomial ring is Cohen-Macaulay, and Huneke's proof [Hu1] about integral closures of the powers of an ideal, and from these standing

proofs Hochster and Huneke pulled the essential ingredients to define a new notion. Not only were they then able to reprove all the mentioned theorems more quickly, but they were able to easily prove greater generalizations, and many new theorems as well. Among the first new theorems were new versions of the Briançon-Skoda theorem, and new results on the Cohen-Macaulayness of direct summands of regular rings, the vanishings of some Tor maps, etc. The new notion was worthy of a name, and Hochster and Huneke called it tight closure.

Tight closure continues to be a good tool in commutative algebra as well as in algebraic geometry. Hochster and Huneke themselves have published several hundred pages on tight closure, all the while developing beautiful commutative algebra. The theory of tight closure has also grown due to the work of Ian Aberbach, Nobuo Hara, Mordechai Katzman, Gennady Lyubeznik, Anurag Singh, Karen Smith, Kei-Ichi Watanabe, and other people. My early research was also in tight closure, as a result of which I became interested in asymptotic properties of powers of ideals. My work on asymptotic properties uses the techniques of tight closure, but not necessarily involve the statements about tight closure.

The extensive tight closure bibliography at the end of these notes is meant to facilitate finding more details and further information. But most of the material for these notes is taken from [HH2], [HH4], [HH9], [HH14], [HH15], [Hu3] and [Sm1], and with major input also from [ElSm], [Ho1], [Hu2], [K1], [Ku1], [Ku2], [Sm13]. The sections of these notes roughly follow my lectures, although some sections took more than one lecture. In particular, the long section onn test elements (Section 6) took about 2 lectures.

I thank IPM for hosting these talks. I thank the many participants who came through snow and dense traffic, made comments, asked questions, explained to me their work, asked for more... My stay in Iran was made very

pleasant and worthwhile by all of them. I especially thank Kamran Divaani-Aazar and Siamak Yassemi for organizing the workshop and for organizing my mathematical activities in Iran, as well as for introducing me to non-mathematical Iran. The ten days of my lectures on tight closure were interspersed with talks on commutative algebra by Rahim Zaare-Nahandi, Kamran Divaani-Aazar, Kazem Khashyarmanesh, Hassan Haghighi, Leila Khatami, Javad Asadollahi, and Tirdad Sharif, and I thank them all for their nice talks. I also thank Mohamad-Taghi Dibaei, Kamran Divaani-Aazar, Hassan Haghighi, Leila Khatami, Tirdad Sharif, Siamak Yassemi, and Hossein Zakeri for their participation in the workshop, and for their personal warmth. I am grateful to Craig Huneke and Melvin Hochster for teaching me about tight closure, and to Steve and Simon Swanson for putting up with my yet another absence. Finally, many thanks go also to Kamran Divaani-Aazar and Siamak Yassemi for the helpful feedback and proofreading of these notes.