# 1 Introduction

One of the major advances of science in the 20th century was the discovery of a mathematical formulation of quantum mechanics by Heisenberg in 1925 [41]. From a mathematical point of view, transition from classical mechanics to quantum mechanics amounts to, among other things, passing from the *commutative algebra* of *classical observables* to the *noncommutative algebra* of *quantum mechanical observables*. Recall that in classical mechanics an observable (e.g. energy, position, momentum, etc.) is a function on a manifold called the phase space of the system. Immediately after Heisenberg's work, ensuing papers by Dirac [32] and Born-Heisenberg-Jordan [5], made it clear that a quantum mechanical observable is a (selfadjoint) operator on a Hilbert space called the state space of the system. Thus the commutative algebra of functions on a space is replaced by the noncommutative algebra of operators on a Hilbert space.

A little more than fifty years after these developments, Alain Connes realized that a similar procedure can in fact be applied to areas of mathematics where the classical notions of space (e.g. measure space, locally compact space, or a smooth space) looses its applicability and pertinence and can be replaced by a new idea of space, represented by a noncommutative algebra.

Connes' theory, which is generally known as *noncommutative geometry*, is a rapidly growing new area of mathematics that interacts with and contributes to many disciplines in mathematics and physics. For a recent survey see Connes' article [16]. Examples of such interactions and contributions include: theory of operator algebras, index theory of elliptic operators, algebraic and differential topology, number theory, standard model of elementary particles, quantum Hall effect, renormalization in quantum field theory, and string theory. (For a description of these relations in more details see the report below.) To understand the basic ideas of noncommutative geometry one should perhaps first come to grips with the idea of a *noncommutative space*.

The inadequacy of the classical notions of space manifests itself for example when one deals with highly singular "bad quotients"; spaces such as the quotient of a nice space by the ergodic action of a group or the space of leaves of a foliation. In all these examples the quotient space is typically ill behaved even as a topological space. For example it may fail to be even Hausdorff, or have enough open sets, let alone being a reasonably smooth space. The unitary dual of a noncompact (Lie) group, except when the group is abelian or almost abelian, is another example of an ill behaved space.

One of Connes' key observations is that in all these situations one can attach a noncommutative algebra, through a *noncommutative quotient construction*, that captures most of the information. Examples of this noncommutative quotient construction include crossed product by action of a group, or by action of a groupoid. In general the noncommutative quotient is the groupoid algebra of a topological groupoid.

Noncommutative geometry has as its limiting case the classical geometry, but geometry expressed in algebraic terms. Thus to understand its relation with classical geometry one should first understand one of the most important ideas of mathematics which can be expressed as a *duality* between commutative algebra and geometry. This is by no means a new observation or a new trend. To the contrary, this duality has always existed and been utilized in mathematics and its applications. The earliest example is perhaps the use of numbers in counting! It is, however, the case that throughout the history each new generation of mathematicians find new ways of formulating this principle and at the same time broaden its scope. Just to mention a few highlights of this rich history we mention Descartes (analytic geometry), Hilbert (affine varieties and commutative algebras), Gelfand-Naimark (locally compact spaces and commutative $C^*$-algebras), Grothendieck (Schemes and topos theory), and Connes (noncommutative geometry).

A key idea here is the well-known relation between a space and the commutative algebra of functions on that space. More precisely there is a duality between certain categories of geometric spaces and categories of algebras representing those spaces. Noncommutative geometry builds on, and vastly extends, this fundamental duality between geometry and commutative algebras.

For example, by a celebrated theorem of Gelfand and Naimark [35] one knows that the category of locally compact Hausdorff spaces is equivalent to the dual of the category of commutative $C^*$-algebras. Thus one can think of not necessarily commutative $C^*$-algebras as the dual of a category of *noncommutative locally compact spaces*. What makes this a successful proposal is first of all a rich supply of examples and secondly the possibility of extending many of the topological and geometric invariants to this new class of spaces. Let us briefly recall a few other examples from a long list of results in mathematics that put in duality certain categories of geometric objects with a corresponding category of algebraic objects.

To wit, Hilbert's Nullstellensatz states that the category of algebraic varieties over an algebraically closed field is equivalent to the dual of the category of finitely generated commutative algebras without nilpotent elements (so called reduced algebras). This is a perfect analogue of the Gelfand-Naimark theorem in the world of commutative algebras.

Similarly, the Serre-Swan theorem states that the category of vector bundles over a compact Hausdorff space (resp. affine algebraic variety) X is equivalent to the category of finitely generated projective modules over the algebra of continuous functions (resp. regular functions) on X.

Thus a pervasive idea in noncommutative geometry is to treat (certain classes) of noncommutative algebras as noncommutative spaces and try to extend tools of geometry, topology, and analysis to this new setting. It should be emphasized, however, that, as a rule, this extension is never straightforward and always involve surprises and new phenomena. For example the theory of the flow of weights and the corresponding modular automorphism group in von Neumann algebras has no counterpart in classical measure theory, though the theory of von Neumann algebras is generally regarded as noncommutative measure theory. Similarly the extension of de Rham homology for manifolds to cyclic cohomology for noncommutative algebras was not straightforward and needed some highly nontrivial considerations.

Of all the topological invariants for spaces, topological $K$-theory has the most straightforward extension to the noncommutative realm. Recall that topological $K$-theory classifies vector bundles on a topological space. Using the above mentioned Serre-Swan theorem, it is natural to define, for a not necessarily commutative ring $A$, $K_0(A)$ as the group defined by the semigroup of isomorphism classes of finite projective $A$-modules. The definition of $K_1(A)$ follows the same pattern as in the commutative case, provided $A$ is a Banach algebra and the main theorem of topological $K$-theory, the Bott periodicity theorem, extends to all Banach algebras.

The situation is much less clear for $K$-homology, the theory dual to $K$-theory. By the work of Atiyah, Brown-Douglas-Fillmore, and Kasparov, one can say, roughly speaking, that $K$-homology cycles on a space $X$ are represented by abstract elliptic operators on $X$ and while $K$-theory classifies vector bundles on $X$, $K$-homology classifies the abstract elliptic operators on $X$. The pairing between $K$-theory and $K$-homology takes the form $<[D],[E]>=$ the Fredholm index of the elliptic operator $D$ with coefficients in the vector bundle $E$. Now one good thing about this way of formulating $K$-homology is that it almost immediately extends to noncommutative $C^*$-algebras. The two theories are unified in a single theory called $KK$-theory due to G. Kasparov.

Cyclic cohomology was discovered by Connes in 1981 [11, 13] as the right noncommutative analogue of de Rham homology of currents and as a target space for noncommutative Chern character maps from both $K$-theory and $K$-homology. One of the main motivations of Connes seems to be transverse index theory on foliated spaces. Cyclic cohomology can be used to identify the $K$-theoretic index of transversally elliptic operators which lie in the $K$-theory of the noncommutative algebra of the foliation. The formalism of cyclic cohomology and Chern-Connes character maps form an indispensable part of noncommutative geometry. In a different direction, cyclic homology also appeared in the 1983 work of Tsygan [60] and was used, independently, also by Loday and Quillen [54] in their study of the Lie algebra homology of the Lie algebra of stable matrices over an associative algebra. We won't pursue this aspect of cyclic homology in these notes.

A very interesting recent develpoment in cyclic cohomology theory is the *Hopf-cyclic cohomology* of Hopf algebras and Hopf module (co)algebras in general. Motivated by the original work of Connes and Moscovici [18, 19] this theory is now extended and elaborated on by several authors [1, 2, 38, 39, 49, 50, 51, 52]. There are also very interesting relations between cocycles for Hopf-cyclic cohomology theory of the Connes-Moscovici Hopf algebra $\mathcal{H}_1$ and operations on spaces of modular forms and modular Hecke algebras [20, 21], and spaces of $\mathbb{Q}$-lattices [23]. We will say nothing about these develpments in these notes. Neither we shall discuss the approach of Cuntz and Quillen to cyclic cohomology theory and their cellebrated proof of excision property for periodic (bivariant) cyclic cohomology [25, 30, 27, 28, 29].

The following "dictionary" illustrates noncommutative analogues of some of the classical theories and concepts originally conceived for spaces. In these notes we deal only with a few items of this dictionary. For a much fuller account and explanations, as well as applications of noncommutative geometry, the reader should consult Connes' beautiful book [15].

| **commutative** | **noncommutative** |
|---|---|
| measure space | von Neumann algebra |
| locally compact space | $C^*$ - algebra |
| vector bundle | finite projective module |
| complex variable | operator on a Hilbert space |
| real variable | sefadjoint operator |
| infinitesimal | compact operator |
| range of a function | spectrum of an operator |
| $K$-theory | $K$-theory |
| vector field | derivation |
| integral | trace |
| closed de Rham current | cyclic cocycle |
| de Rham complex | Hochschild homology |
| de Rham cohomology | cyclic homolgy |
| Chern character | Chern-Connes character |
| Chern-Weil thoery | noncommutative Chern-Weil thoery |
| elliptic operator | $K$-cycle |
| spin Riemannian manifold | spectral triple |
| index theorem | local index formula |
| group, Lie algebra | Hopf algebra, quantum group |
| symmetry | action of Hopf algebra |

Noncommutative geometry is already a vast subject. These notes are just meant to be an introduction to a few aspects of this fascinating enterprize. To get a much better sense of the beauty and depth of the subject the reader should consult Connes' magnificent book [15] or his recent survey [16] and references therein. Meanwhile, to give a sense of the state of the subject at the present time, its relation with other fields of mathematics, and its most pressing issues, we reproduce here part of the text of the final report prepared by the organizers of a conference on noncommutative geometry in 2003 [1]:

### "1. The Baum-Connes conjecture

*This conjecture, in its simplest form, is formulated for any locally compact topological group. There are more general Baum-Connes conjectures*

[1]BIRS Workshop on Noncommutative Geometry, Banff International Research Station, Banff, Alberta, Canada, April 2003, Organized by Alain Connes, Joachim Cuntz, George Elliott, Masoud Khalkhali, and Boris Tsygan. Full report available at: www.pims.math.ca/birs.

*with coefficients for groups acting on $C^*$-algebras, for groupoid $C^*$-algebras, etc., that for the sake of brevity we don't consider here. In a nutshell the Baum-Connes conjecture predicts that the K-theory of the group $C^*$-algebra of a given topological group is isomorphic, via an explicit map called the Baum-Connes map, to an appropriately defined K-homology of the classifying space of the group. In other words invariants of groups defined through noncommutative geometric tools coincide with invariants defined through classical algebraic topology tools. The Novikov conjecture on the homotopy invariance of higher signatures of non-simply connected manifolds is a consequence of the Baum-Connes conjecture (the relevant group here is the fundamental group of the manifold). Major advances were made in this problem in the past seven years by Higson-Kasparov, Lafforgue, Nest-Echterhoff-Chabert, Yu, Puschnigg and others.*

## 2. Cyclic cohomology and KK-theory

*A major discovery made by Alain Connes in 1981, and independently by Boris Tsygan in 1983, was the discovery of cyclic cohomology as the right noncommutative analogue of de Rham homology and a natural target for a Chern character map from K-theory and K-homology. Coupled with K-theory, K-homology and KK-theory, the formalism of cyclic cohomology fully extends many aspects of classical differential topology like Chern-Weil theory to noncommutative spaces. It is an indispensable tool in noncommutative geometry. In recent years Joachim Cuntz and Dan Quillen have formulated an alternative powerful new approach to cyclic homology theories which brings with it many new insights as well as a successful resolution of an old open problem in this area, namely establishing the excision property of periodic cyclic cohomology.*

*For applications of noncommutative geometry to problems of index theory, e.g. index theory on foliated spaces, it is necessary to extend the formalism of cyclic cohomology to a bivariant cyclic theory for topological algebras and to extend Connes's Chern character to a fully bivariant setting. The most general approach to this problem is due to Joachim Cuntz. In fact the approach of Cuntz made it possible to extend the domain (and definition) of KK-theory to very general categories of topological algebras (rather than just $C^*$-algebras). The fruitfulness of this idea manifests itself in the V. Lafforgue's proof of the Baum-Connes conjecture for groups with property T, where the extension of KK functor to Banach algebras plays an important role.*

*A new trend in cyclic cohomology theory is the study of the cyclic coho-*

*mology of Hopf algebras and quantum groups. Many noncommutative spaces, such as quantum spheres and quantum homogeneous spaces, admit a quantum group of symmetries. A remarkable discovery of Connes and Moscovici in the past few years is the fact that diverse structures, such as the space of leaves of a (codimension one) foliation or the space of modular forms, have a unified quantum symmetry. In their study of transversally elliptic operators on foliated manifolds Connes and Moscovici came up with a new noncommutative and non-cocommutative Hopf algebra denoted by $\mathcal{H}_n$ (the Connes-Moscovici Hopf algebra). $\mathcal{H}_n$ acts on the transverse foliation algebra of codimension $n$ foliations and thus appears as the quantized symmetries of a foliation. They noticed that if one extends the noncommutative Chern-Weil theory of Connes from group and Lie algebra actions to actions of Hopf algebras, then the characteristic classes defined via the local index formula are in the image of this new characteristic map. This extension of Chern-Weil theory involved the introduction of cyclic cohomology for Hopf algebras.*

## 3. Index theory and noncommutative geometry

*The index theorem of Atiyah and Singer and its various generalizations and ramifications are at the core of noncommutative geometry and its applications. A modern abstract index theorem in the noncommutative setting is the local index formula of Connes and Moscovici. A key ingredient of such an abstract index formula is the idea of an spectral triple due to Connes. Broadly speaking, and neglecting the parity, a spectral triple $(A, H, D)$ consists of an algebra $A$ acting by bounded operators on the Hilbert space $H$ and a self-adjoint operator $D$ on $H$. This data must satisfy certain regularity properties which constitute an abstraction of basic elliptic estimates for elliptic PDE's acting on sections of vector bundles on compact manifolds. The local index formula replaces the old non-local Chern-Connes cocycle by a new Chern character form $Ch(A, H, D)$ of the given spectral triple in the cyclic complex of the algebra $A$. It is a local formula in the sense that the cochain $Ch(A, H, D)$ depends, in the classical case, only on the germ of the heat kernel of $D$ along the diagonal and in particular is independent of smooth perturbations. This makes the formula extremely attractive for practical calculations. The challenge now is to apply this formula to diverse situations beyond the cases considered so far, namely transversally elliptic operators on foliations (Connes and Moscovici) and the Dirac operator on quantum $SU_2$ (Connes).*

## 4. Noncommutative geometry and number theory

*Current applications and connections of noncommutative geometry to number theory can be divided into four categories. (1) The work of Bost and Connes, where they construct a noncommutative dynamical system $(B, \sigma_t)$ with partition function the Riemann zeta function $\zeta(\beta)$, where $\beta$ is the inverse temperature. They show that at the pole $\beta = 1$ there is an spontaneous symmetry breaking. The symmetry group of this system is the group of idéles which is isomorphic to the Galois group $Gal(Q^{ab}/Q)$. This gives a natural interpretation of the zeta function as the partition function of a quantum statistical mechanical system. In particular the class field theory isomorphism appears very naturally in this context. This approach has been extended to the Dedekind zeta function of an arbitrary number field by Cohen, Harari-Leichtnam, and Arledge-Raeburn-Laca. All these results concern abelian extensions of number fields and their generalization to non-abelian extensions is still lacking. (2) The work of Connes on the Riemann hypothesis. It starts by producing a conjectural trace formula which refines the Arthur-Selberg trace formula. The main result of this theory states that this trace formula is valid if and only if the Riemann hypothesis is satisfied by all L-functions with Grössencharakter on the given number field $k$. (3) The work of Connes and Moscovici on quantum symmetries of the modular Hecke algebras $\mathcal{A}(\Gamma)$ where they show that this algebra admits a natural action of the transverse Hopf algebra $\mathcal{H}_1$. Here $\Gamma$ is a congruence subgroup of $SL(2, Z)$ and the algebra $\mathcal{A}(\Gamma)$ is the crossed product of the algebra of modular forms of level $\Gamma$ by the action of the Hecke operators. The action of the generators $X, Y$ and $\delta_n$ of $\mathcal{H}_1$ corresponds to the Ramanujan operator, to the weight or number operator, and to the action of certain group cocycles on $GL^+(2, Q)$, respectively. What is very surprising is that the same Hopf algebra $\mathcal{H}_1$ also acts naturally on the (noncommutative) transverse space of codimension one foliations. (4) Relations with arithmetic algebraic geometry and Arakelov theory. This is currently being pursued by Consani, Deninger, Manin, Marcolli and others.*

## 5. Deformation quantization and quantum geometry

*The noncommutative algebras that appear in noncommutative geometry usually are obtained either as the result of a process called noncommutative quotient construction or by deformation quantization of some algebra of functions on a classical space. These two constructions are not mutually exclusive. The starting point of deformation quantization is an algebra* *of functions on a Poisson manifold where the Poisson structure gives the infinitesimal direction of quantization. The existence of deformation quantizations for all Poisson manifolds was finally settled by M. Kontsevich in 1997 after a series of partial results for symplectic manifolds. The algebra of pseudodifferential operators on a manifold is a deformation quantization of the algebra of classical symbols on the cosphere bundle of the manifold. This simple observation is the beginning of an approach to the proof of the index theorem, and its many generalizations by Elliott-Natsume-Nest and Nest-Tsygan, using cyclic cohomology theory. The same can be said about Connes's groupoid approach to index theorems. In a different direction, quantum geometry also consists of the study of noncommutative metric spaces and noncommutative complex structures."*

Let us now briefly describe the contents of these notes. In Section 2 we describe some of the fundamental algebra-geometry correspondences at work in mathematics. The most basic ones for noncommutative geometry are the Gelfand-Naimark and the Serre-Swan theorems. In Section 3 we describe the noncommutative quotient construction and give several examples. This is one of the most universal methods of constructing noncommutative spaces directly related to classical geometric examples. Section 4 is devoted to cyclic cohomology and its various definitions. In Section 5 we define the Chern-Connes character map, or the noncommutative Chern character map, from $K$-theory to cyclic cohomology. In an effort to make these notes as self contained as possible, we have added three appendices covering very basic material on $C^*$-algebras, projective modules, and category theory language.

These notes are partly based on series of lectures I gave at the Fields Institute in Toronto, Canada, in Fall 2002 and at the Institute for Studies in Theoretical Physics and Mathematics (IPM), Tehran, Iran, in Spring 2004. I also used part of these notes in my lectures at the *Second Annual Spring Institute and Workshop on Noncommutative Geometry* in Spring 2004, Vanderbilt University, USA. It is a great pleasure to thank the organizers of this event, Alain Connes (director), to whom I owe much more than I can adequately express, Dietmar Bisch, Bruce Hughes, Gennady Kasparov, and Guoliang Yu. I would also like to thank G.B. Khosrovshahi (Reza) the Head of School of Mathematics of IPM in Tehran whose encouragement and support was instrumental in bringing these notes to existence.